

COERCIVITY CONDITION FOR HIGHER ORDER MOMENTS FOR NONLINEAR SPDES AND EXISTENCE OF SOLUTION UNDER LOCAL MONOTONICITY

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ABSTRACT. Higher order moment estimates for solutions to nonlinear SPDEs governed by locally-monotone operators are obtained under appropriate coercivity condition. These are then used to extend known existence and uniqueness results for nonlinear SPDEs under local monotonicity conditions to allow derivatives in the operator acting on the solution under the stochastic integral.

1. INTRODUCTION

Let $T > 0$ be given, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a stochastic basis and $W := (W_t)_{t \in [0, T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t \in [0, T]}$, i.e. the coordinate processes $(W_t^j)_{t \in [0, T]}, j \in \mathbb{N}$ are independent \mathcal{F}_t -adapted Wiener processes and $W_t - W_s$ is independent of \mathcal{F}_s for $s \leq t$. Further assume that H is a separable Hilbert space, V is a separable, reflexive Banach space embedded continuously and densely in H and V^* is the dual of V . Identifying H with H^* using the Riesz representation and the inner product in H one obtains the Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

where \hookrightarrow denotes continuous and dense embeddings.

Consider the stochastic evolution equation

$$u_t = u_0 + \int_0^t A_s(u_s)ds + \sum_{j=1}^{\infty} \int_0^t B_s^j(u_s)dW_s^j, \quad t \in [0, T], \quad (1.1)$$

where the initial condition u_0 is an H -valued \mathcal{F}_0 -measurable random variable. Moreover A and $B^j, j \in \mathbb{N}$, are progressively measurable non-linear operators mapping $[0, T] \times \Omega \times V$ into V^* and H respectively. The exact assumptions will be given in Section 2.

The nonlinear stochastic evolution equation (1.1) has been initially studied in Pardoux [12] and Krylov and Rozovskii [8], where a priori estimates are proved, giving the second moment estimates under what are now classical monotonicity, coercivity and growth assumptions. This then allows the authors to obtain existence and uniqueness of solutions to (1.1). One of the key results in [8] is the theorem about Itô formula for the square of the norm of a continuous semimartingale in a Gelfand triple obtained separately

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from the related stochastic evolution equation. This theorem provides the continuity of the solution in the pivot space of the Gelfand triple and is key to obtaining the a priori estimates and in proving the existence and uniqueness of the solution. These, now classical results, have been generalized in a number of directions. Of those one notes the inclusion of general càdlàg semi-martingales as the driving process in stochastic integral, see Gyöngy and Krylov [5] and Gyöngy [4]. Closely related to the results in this paper is the work by Liu and Röckner [10]. They extended the framework of Krylov and Rozovskii [8] to stochastic evolution equations when the operators are only locally monotone and the operator A , which is the operator acting in the bounded variation term, satisfies a less restrictive growth condition. To obtain a generalization in this direction Liu and Röckner [10] need higher order moment estimates and to obtain them they place a restrictive assumption on the growth of the operator B , which is the operator acting on the solution under the stochastic integral. As a consequence one may not have derivatives appearing in this operator. The local monotonicity and coercivity conditions are further weakened in Liu and Röckner [11] but again at the expense of having a growth restriction on the operator B . Moreover, Brzeźniak, Liu and Zhu [2] extend the results in [10] to include equations driven by Lévy noise but again with growth restrictions on the operators appearing under the stochastic integrals. Fully deterministic equations under local monotonicity assumptions are considered in Liu [9].

The main contribution of this paper is to identify appropriate coercivity assumption which allows one to obtain higher order moment estimates and to prove existence and uniqueness of solutions to (1.1) without the need to explicitly restrict the growth of the operator B . Examples of stochastic partial differential equations for which existence and uniqueness follows from neither [8] nor [10] are given. Finally, an example is considered that, together with results from Brzeźniak and Veraar [3], shows that the coercivity assumption identified in this paper is a natural one.

This article is organized as follows. In Section 2 the main results about higher-order moment estimates as well as existence and uniqueness of solutions are stated, together with the assumptions required. In Section 3 some auxiliary lemmas are presented and proved. Section 4 is devoted to proving the a priori estimates and uniqueness of the solution. Galerkin discretization is used to obtain a finite-dimensional approximation to (1.1) in Section 5. Moreover moment bounds for the solutions of the finite-dimensional equations, uniform in the discretization parameter, are established. These are used in Section 6 to prove existence of solution to (1.1). Finally, Section 7 is devoted to examples of stochastic partial differential equations which fit into the framework of this article.

2. ASSUMPTIONS AND MAIN RESULTS

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e., the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, \mathcal{F}_0 contains all the \mathbb{P} -null sets that are in \mathcal{F} and $(\mathcal{F}_t)_{t \in [0, T]}$ is right continuous. Let $W := (W_t)_{t \in [0, T]}$ be an infinite dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t \in [0, T]}$.

Let $(X, |\cdot|_X)$ be a separable and reflexive Banach Space. For a given constant $p \in [1, \infty)$, $L^p(\Omega; X)$ denotes the Bochner–Lebesgue space of equivalence classes of random variables x taking values in X such that the norm

$$|x|_{L^p(\Omega; X)} := (\mathbb{E}|x|_X^p)^{\frac{1}{p}}$$

is finite. Again, $L^p(0, T; X)$ denotes the Bochner–Lebesgue space of equivalence classes of X -valued measurable functions such that the norm

$$|x|_{L^p(0, T; X)} := \left(\int_0^T |x_t|_X^p dt \right)^{\frac{1}{p}}$$

is finite while $L^\infty(0, T; X)$ denotes the Bochner–Lebesgue space of X -valued measurable functions which are essentially bounded, i.e.

$$|x|_{L^\infty(0, T; X)} := \operatorname{ess\,sup}_{t \in (0, T)} |x_t|_X < \infty.$$

Finally, $L^p((0, T) \times \Omega; X)$ denotes the Bochner–Lebesgue space of equivalence classes of X -valued stochastic processes which are progressively measurable and the norm

$$|x|_{L^p((0, T) \times \Omega; X)} := \left(\mathbb{E} \int_0^T |x_t|_X^p dt \right)^{\frac{1}{p}}$$

is finite.

Moreover, let $(H, (\cdot, \cdot), |\cdot|_H)$ be a separable Hilbert space, identified with its dual and let $(V, |\cdot|_V)$ denote a separable, reflexive Banach space embedded continuously and densely in H with $(V^*, |\cdot|_{V^*})$ denoting its dual and $\langle \cdot, \cdot \rangle$ the duality pairing between V and V^* . Thus one has

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with continuous and dense embeddings.

Let A and B^j , $j \in \mathbb{N}$, be non-linear operators mapping $[0, T] \times \Omega \times V$ into V^* and H respectively. Assume that for all $v, w \in V$, the processes $(\langle A_t(v), w \rangle)_{t \in [0, T]}$ and $((B_t^j(v), w))_{t \in [0, T]}$ are progressively measurable. Since the concept of weak measurability and strong measurability of a mapping coincide if the codomain is separable, one gets that for all $v \in V$, $j \in \mathbb{N}$, $(A_t(v))_{t \in [0, T]}$ and $(B_t^j(v))_{t \in [0, T]}$ are progressively measurable. Finally, u_0 is assumed to be a given H -valued \mathcal{F}_0 -measurable random variable.

The following assumptions are made on the operators. There exist constants $\alpha > 1$, $\beta \geq 0$, $p_0 \geq \beta + 2$, $\theta > 0$, K, L and a nonnegative $f \in L^{\frac{p_0}{2}}((0, T) \times \Omega; \mathbb{R})$ such that, almost surely, the following conditions hold for all $t \in [0, T]$.

A - 1 (Hemicontinuity). *For all $y, x, \bar{x} \in V$, the map*

$$\varepsilon \mapsto \langle A_t(x + \varepsilon \bar{x}), y \rangle$$

is continuous.

A - 2 (Local Monotonicity). *For all $x, \bar{x} \in V$,*

$$\begin{aligned} 2\langle A_t(x) - A_t(\bar{x}), x - \bar{x} \rangle + \sum_{j=1}^{\infty} |B_t^j(x) - B_t^j(\bar{x})|_H^2 \\ \leq L(1 + |\bar{x}|_V^\alpha)(1 + |\bar{x}|_H^\beta)|x - \bar{x}|_H^2. \end{aligned}$$

A - 3 (Coercivity). For all $x, \bar{x} \in V$,

$$2\langle A_t(x), x \rangle + (p_0 - 1) \sum_{j=1}^{\infty} |B_t^j(x)|_H^2 + \theta |x|_V^\alpha \leq f_t + K |x|_H^2.$$

A - 4 (Growth of A). For all $x \in V$,

$$|A_t(x)|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (f_t + K |x|_V^\alpha)(1 + |x|_H^\beta).$$

Note that, if $p_0 = 2$, i.e. $\beta = 0$ and $L = 0$, then the conditions A-1 to A-4 reduce to corresponding ones used in Krylov and Rozovskii [8].

Throughout the article a generic constant C will be used and it may change from line to line.

Remark 2.1. From Assumptions A-3 and A-4, one obtains

$$\sum_{j=1}^{\infty} |B_t^j(x)|_H^2 \leq C(1 + f_t^{\frac{p_0}{2}} + |x|_H^{p_0} + |x|_V^\alpha + |x|_V^\alpha |x|_H^\beta)$$

almost surely for all $t \in [0, T]$ and $x \in V$. Indeed, using Cauchy-Schwartz inequality, Young's inequality and Assumption A-4, one obtains that almost surely for all $x \in V$ and $t \in [0, T]$,

$$\begin{aligned} |\langle A_t(x), x \rangle| &\leq \frac{\alpha-1}{\alpha} |A_t(x)|_{V^*}^{\frac{\alpha}{\alpha-1}} + \frac{1}{\alpha} |x|_V^\alpha \\ &\leq \frac{\alpha-1}{\alpha} ((f_t + K |x|_V^\alpha)(1 + |x|_H^\beta)) + \frac{1}{\alpha} |x|_V^\alpha \\ &\leq C(f_t + |x|_V^\alpha + |x|_V^\alpha |x|_H^\beta + f_t^{\frac{p_0}{2}} + (1 + |x|_H^{p_0})). \end{aligned}$$

The above inequality along with Assumption A-3 gives the result.

Remark 2.2. From Assumptions A-1, A-2 and A-4 one obtains that almost surely for all $t \in [0, T]$, the operator A_t is demicontinuous, i.e. $v_n \rightarrow v$ in V implies that $A_t(v_n) \rightharpoonup A_t(v)$ in V^* . This follows using similar arguments as in the proof of Lemma 2.1 in Krylov and Rozovskii [8].

One consequence of this remark is that, progressive measurability of some process $(v_t)_{t \in [0, T]}$ implies the progressive measurability of the process $(A_t(v_t))_{t \in [0, T]}$.

Definition 2.3 (Solution). An adapted, continuous, H -valued process u is called a solution of the stochastic evolution equation (1.1) if

i) $dt \times \mathbb{P}$ almost everywhere $u \in V$ and

$$\mathbb{E} \int_0^T (|u_t|_V^\alpha + |u_t|_H^2) dt < \infty,$$

ii) almost surely

$$\int_0^T (|u_t|_H^\beta + |u_t|_V^\alpha |u_t|_H^\beta) dt < \infty,$$

iii) for every $t \in [0, T]$ and $\phi \in V$

$$(u_t, \phi) = (u_0, \phi) + \int_0^t \langle A_s(u_s), \phi \rangle ds + \sum_{j=1}^{\infty} \int_0^t (\phi, B_s^j(u_s)) dW_s^j \quad a.s.$$

The following are the main results of this article.

Theorem 2.4 (A priori estimates). *If u is a solution of (1.1) and Assumptions A-3 and A-4 hold, then*

$$\begin{aligned} \sup_{t \in [0, T]} \mathbb{E} |u_t|_H^{p_0} + \mathbb{E} \int_0^T |u_t|_H^{p_0-2} |u_t|_V^\alpha dt + \mathbb{E} \int_0^T |u_t|_V^\alpha dt \\ \leq C \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right). \end{aligned} \quad (2.1)$$

Moreover,

$$\mathbb{E} \sup_{t \in [0, T]} |u_t|_H^p \leq C \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right), \quad (2.2)$$

with $p = 2$ in case $p_0 = 2$ and with any $p \in [2, p_0)$ in case $p_0 > 2$, where C depends only on p_0, K, T and θ .

Theorem 2.5 (Uniqueness of solution). *Let Assumptions A-2 to A-4 hold and $u_0 \in L^{p_0}(\Omega; H)$. If u and \bar{u} are two solutions of (1.1), then the processes u and \bar{u} are indistinguishable, i.e.*

$$\mathbb{P} \left(\sup_{t \in [0, T]} |u_t - \bar{u}_t|_H = 0 \right) = 1.$$

Theorem 2.6 (Existence of solution). *If Assumptions A-1 to A-4 hold and $u_0 \in L^{p_0}(\Omega; H)$, then the stochastic evolution equation (1.1) has a unique solution.*

3. AUXILIARY RESULTS

The following two lemmas are not new but they will be needed in what follows and are included for the convenience of the reader. The first lemma is a simpler version of an inequality of Lenglart, see, e.g. Gyöngy and Krylov [6].

Lemma 3.1. *Let f be a real-valued, non-negative, adapted, continuous process such that there is constant $K > 0$ so that*

$$\mathbb{E} f_\tau \leq K$$

for any bounded stopping time τ . Then for any $r \in (0, 1)$, and any bounded stopping time τ one has

$$\mathbb{E} \sup_{t \leq \tau} f_t^r \leq \frac{r}{1-r} K.$$

Proof. For any $c \geq 0$, define

$$\theta_f := \inf \{ t \geq 0 : f_t \geq c \}.$$

Note that $\theta_f \leq \tau$ implies $f_{\tau \wedge \theta_f} = f_{\theta_f} = c$ and therefore Markov's inequality leads one to

$$\mathbb{P} \left(\sup_{t \leq \tau} f_t \geq c \right) = \mathbb{P}(\theta_f \leq \tau) \leq \mathbb{P}(f_{\tau \wedge \theta_f} \geq c) \leq \frac{1}{c} \mathbb{E}[f_{\tau \wedge \theta_f}] \leq \frac{K}{c}.$$

Replacing c by $c^{\frac{1}{r}}$ with $r \in (0, 1)$, one obtains

$$\mathbb{P}(\sup_{t \leq \tau} f_t^r > c) \leq K c^{-\frac{1}{r}} \quad (3.1)$$

Defining $Y := \sup_{t \leq \tau} f_t^r$ and integrating by parts, one obtains

$$\int_0^\infty \mathbb{P}(Y > c) dc = \int_0^\infty c d\mathbb{P}(Y \leq c) = \mathbb{E}[Y].$$

Therefore, one obtains the result integrating from 0 to ∞ in (3.1). \square

The second auxiliary lemma allows one to obtain weakly-star convergent subsequences, under appropriate assumptions.

Lemma 3.2. *Let X be a separable Banach space with dual X^* and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^* . If (S, Σ, μ) is a measure space with $\mu(S) < \infty$, and $(u_n)_{n \in \mathbb{N}}$ is a sequence satisfying*

$$\sup_n \int_S |u_n|_{X^*}^p d\mu < \infty \quad (3.2)$$

for some $p \geq 2$, then there exists a subsequence (n_k) and $u \in L^p(S, X^*)$ such that (u_{n_k}) converges weakly-star to u as $n_k \rightarrow \infty$, i.e.,

$$\int_S \langle u_{n_k}, \varphi \rangle d\mu \rightarrow \int_S \langle u, \varphi \rangle d\mu \quad \forall \varphi \in L^{\frac{p}{p-1}}(S, X).$$

Proof. Let $(\phi_i)_{i \in \mathbb{N}}$ be dense subset in X . Then, it is sufficient to show

$$\int_S \langle u_{n_k}, \phi_i \rangle \psi d\mu \rightarrow \int_S \langle u, \phi_i \rangle \psi d\mu \quad \forall i \in \mathbb{N}, \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R})$$

for some subsequence (n_k) and $u \in L^p(S, X^*)$. Observe that, in view of Hölder's inequality and (3.2), we have

$$\int_S |\langle u_n, \phi_i \rangle|^p d\mu \leq \int_S |u_n|_{X^*}^p |\phi_i|_X^p d\mu < C |\phi_i|_X^p$$

for some constant C independent of n . Thus, $\langle u_n, \phi_1 \rangle$ is a uniformly bounded sequence in the reflexive space $L^p(S, \mathbb{R})$. Therefore, there exists a subsequence (n_1) and $\xi_1 \in L^p(S, \mathbb{R})$ such that

$$\int_S \langle u_{n_1}, \phi_1 \rangle \psi d\mu \rightarrow \int_S \xi_1 \psi d\mu \quad \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R}).$$

Repeating the above process with each ϕ_i and subsequence obtained from previous step, there exists a subsequence (n_k) and $(\xi_i)_{i \in \mathbb{N}}$ such that

$$\int_S \langle u_{n_k}, \phi_i \rangle \psi d\mu \rightarrow \int_S \xi_i \psi d\mu \quad \forall i \in \mathbb{N}, \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R}).$$

Finally, one can define $u \in L^p(S, X^*)$ by

$$\int_S \langle u, \phi_i \rangle \psi d\mu := \int_S \xi_i \psi d\mu \quad \forall i \in \mathbb{N}, \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R})$$

and note that,

$$\int_S \langle u_{n_k}, \phi_i \rangle \psi d\mu \rightarrow \int_S \xi_i \psi d\mu = \int_S \langle u, \phi_i \rangle \psi d\mu \quad \forall i \in \mathbb{N}, \forall \psi \in L^{\frac{p}{p-1}}(S, \mathbb{R})$$

as desired. \square

4. A PRIORI ESTIMATES AND UNIQUENESS OF SOLUTION

Proof of Theorem 2.4. Let u be a solution to equation (1.1) in the sense of Definition 2.3. Then, applying the Itô's formula for the square of the norm (see, e.g., Theorem 3.2 in [8] or Theorem 4.2.5 in [13]), one obtains

$$\begin{aligned} |u_t|_H^2 &= |u_0|_H^2 + \int_0^t \left(2\langle A_s(u_s), u_s \rangle + \sum_{j=1}^{\infty} |B_s^j(u_s)|_H^2 \right) ds \\ &\quad + 2 \sum_{j=1}^{\infty} \int_0^t (u_s, B_s^j(u_s)) dW_s^j \end{aligned} \quad (4.1)$$

almost surely for all $t \in [0, T]$. Notice that this is a 1-dimensional Itô process. Thus, by Itô's formula,

$$\begin{aligned} d|u_t|_H^{p_0} &= \frac{p_0}{2} |u_t|_H^{p_0-2} \left(2\langle A_t(u_t), u_t \rangle + \sum_{j=1}^{\infty} |B_t^j(u_t)|_H^2 \right) dt \\ &\quad + p_0 |u_t|_H^{p_0-2} \sum_{j=1}^{\infty} (u_t, B_t^j(u_t)) dW_t^j + \frac{p_0(p_0-2)}{2} |u_t|_H^{p_0-4} \sum_{j=1}^{\infty} |(u_t, B_t^j(u_t))|_H^2 dt \end{aligned}$$

almost surely for all $t \in [0, T]$, which on using Cauchy-Schwarz inequality gives

$$\begin{aligned} d|u_t|_H^{p_0} &\leq \frac{p_0}{2} |u_t|_H^{p_0-2} \left(2\langle A_t(u_t), u_t \rangle + (p_0-1) \sum_{j=1}^{\infty} |B_t^j(u_t)|_H^2 \right) dt \\ &\quad + p_0 |u_t|_H^{p_0-2} \sum_{j=1}^{\infty} (u_t, B_t^j(u_t)) dW_t^j. \end{aligned} \quad (4.2)$$

One aims to apply Lemma 3.1. To that end let τ be some stopping time. Moreover, to estimate the term containing the stochastic integral in (4.2), one needs a sequence $(\sigma_n)_{n \in \mathbb{N}}$ of stopping times converging to T as $n \rightarrow \infty$, defined by

$$\sigma_n := \inf\{t \in [0, T] : |u_t|_H > n\} \wedge T. \quad (4.3)$$

By using Assumption A-3 and Young's inequality in (4.2), one obtains

$$\begin{aligned} |u_{t \wedge \sigma_n \wedge \tau}|_H^{p_0} &\leq |u_0|_H^{p_0} + \frac{p_0}{2} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0-2} \left(f_s + K|u_s|_H^2 - \theta|u_s|_V^\alpha \right) ds \\ &\quad + p_0 \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0-2} (u_s, B_s^j(u_s)) dW_s^j \\ &\leq |u_0|_H^{p_0} + \int_0^{t \wedge \sigma_n \wedge \tau} f_s^{\frac{p_0}{2}} ds + \frac{p_0-2}{2} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0} ds \\ &\quad + \frac{p_0}{2} K \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0} ds - \theta \frac{p_0}{2} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0-2} |u_s|_V^\alpha ds \\ &\quad + p_0 \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0-2} (u_s, B_s^j(u_s)) dW_s^j. \end{aligned}$$

Thus,

$$\begin{aligned}
& |u_{t \wedge \sigma_n \wedge \tau}|_H^{p_0} + \theta \frac{p_0}{2} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0-2} |u_s|_V^\alpha ds \\
& \leq |u_0|_H^{p_0} + \int_0^{t \wedge \sigma_n \wedge \tau} f_s^{\frac{p_0}{2}} ds + \frac{p_0(K+1)-2}{2} \int_0^t \mathbf{1}_{\{s \leq \sigma_n \wedge \tau\}} |u_s|_H^{p_0} ds \\
& \quad + p_0 \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} \mathbf{1}_{\{s \leq \tau\}} |u_s|_H^{p_0-2} (u_s, B_s^j(u_s)) dW_s^j. \tag{4.4}
\end{aligned}$$

Then in view of Remark 2.1 and the fact that u is a solution of equation (1.1), it follows that

$$\mathbb{E} \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} \mathbf{1}_{\{s \leq \tau\}} |u_s|_H^{p_0-2} (u_s, B_s^j(u_s)) dW_s^j = 0.$$

Therefore, taking expectation in (4.4), one obtains

$$\begin{aligned}
& \mathbb{E} |u_{t \wedge \sigma_n \wedge \tau}|_H^{p_0} + \theta \frac{p_0}{2} \mathbb{E} \int_0^{t \wedge \sigma_n \wedge \tau} |u_s|_H^{p_0-2} |u_s|_V^\alpha ds \\
& \leq \mathbb{E} |u_0|_H^{p_0} + \mathbb{E} \int_0^T f_s^{\frac{p_0}{2}} ds + \frac{p_0(K+1)-2}{2} \mathbb{E} \int_0^t |u_{s \wedge \sigma_n \wedge \tau}|_H^{p_0} ds. \tag{4.5}
\end{aligned}$$

From this Gronwall's lemma yields

$$\mathbb{E} |u_{t \wedge \sigma_n \wedge \tau}|_H^{p_0} \leq e^{\frac{p_0(K+1)-2}{2}T} \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right) \tag{4.6}$$

for all $t \in [0, T]$. Letting $n \rightarrow \infty$ and using Fatou's lemma, one obtains

$$\mathbb{E} |u_{t \wedge \tau}|_H^{p_0} \leq e^{\frac{p_0(K+1)-2}{2}T} \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right) \tag{4.7}$$

for all $t \in [0, T]$. Using Lemma 3.1, with $f_t := |u_{T \wedge t}|_H^{p_0}$, one gets

$$\mathbb{E} \sup_{t \in [0, T]} |u_t|_H^{p_0 r} \leq \frac{r}{1-r} e^{\frac{p_0(K+1)-2}{2}T} \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right)$$

for any $r \in (0, 1)$, which proves (2.2) in case $p_0 > 2$.

In order to prove (2.1) the estimate (4.6) is used in the right-hand side of (4.5) with $\tau = T$ and with $n \rightarrow \infty$. One thus obtains

$$\mathbb{E} |u_t|_H^{p_0} + \theta \frac{p_0}{2} \mathbb{E} \int_0^t |u_s|_H^{p_0-2} |u_s|_V^\alpha ds \leq C \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right) \tag{4.8}$$

for all $t \in [0, T]$. If Assumption A-3 holds for some $p_0 \geq \beta + 2$, then it holds for $p_0 = 2$ as well. Thus, using the stopping times $(\sigma_n)_{n \in \mathbb{N}}$ in (4.1) and taking expectation, one obtains, using the same localizing argument as before, that

$$\mathbb{E} |u_t|_H^2 + \theta \mathbb{E} \int_0^t |u_s|_V^\alpha ds \leq \mathbb{E} \left(|u_0|_H^2 + \int_0^T f_s ds \right) + \mathbb{E} \int_0^t K |u_s|_H^2 ds. \tag{4.9}$$

From this, together with (4.8), one can see that

$$\theta \mathbb{E} \int_0^T |u_s|_V^\alpha ds \leq C \mathbb{E} \left(|u_0|_H^{p_0} + \int_0^T f_s^{\frac{p_0}{2}} ds \right)$$

and hence (2.1) holds.

To complete the proof it remains to show (2.2) in case $p_0 = 2$. This is done using the same argument as in Krylov and Rozovskii [8]. It is included here for convenience of the reader. Considering the sequence of stopping times σ_n defined in (4.3) and using Remark 2.1 along with Definition 2.3, one observes that the stochastic integral in the right-hand side of (4.1) is a local martingale. Thus invoking the Burkholder–Davis–Gundy inequality, one gets

$$\mathbb{E} \sup_{t \in [0, T]} \left| \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, B_s^j(u_s)) dW_s^j \right| \leq 4\mathbb{E} \left(\sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |(u_s, B_s^j(u_s))|^2 ds \right)^{\frac{1}{2}}.$$

Further, on using Cauchy–Schwartz inequality, Remark 2.1 and Young’s inequality one obtains

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left| \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, B_s^j(u_s)) dW_s^j \right| &\leq 4\mathbb{E} \left(\sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |u_s|_H^2 |B_s^j(u_s)|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq 4\mathbb{E} \left(\sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 \int_0^{T \wedge \sigma_n} (f_s + |u_s|_H^2 + |u_s|_V^\alpha) ds \right)^{\frac{1}{2}} \\ &\leq \epsilon \mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 + C \mathbb{E} \int_0^{T \wedge \sigma_n} (f_s + |u_s|_H^2 + |u_s|_V^\alpha) ds. \end{aligned} \quad (4.10)$$

Moreover, taking supremum and then expectation in (4.1) and using Assumption A-3 along with (4.10), one obtains

$$\begin{aligned} &\mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 \\ &\leq C \left(\mathbb{E} |u_0|_H^2 + \mathbb{E} \int_0^T f_s ds + \mathbb{E} \int_0^T |u_s|_V^\alpha ds + \sup_{t \in [0, T]} \mathbb{E} |u_t|_H^2 \right) + \epsilon \mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2. \end{aligned}$$

Finally, by choosing ϵ small and using (2.1) for $p_0 = 2$, one obtains

$$\mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 \leq C \left(\mathbb{E} |u_0|_H^2 + \mathbb{E} \int_0^T f_s ds \right)$$

which on allowing $n \rightarrow \infty$ and using Fatou’s lemma finishes the proof. \square

Definition 4.1. Let Ψ be defined as the collection of V -valued and \mathcal{F}_t -adapted processes ψ satisfying

$$\int_0^T \rho(\psi_s) ds < \infty \quad \text{a.s.},$$

where

$$\rho(x) := L(1 + |x|_V^\alpha)(1 + |x|_H^\beta) \quad (4.11)$$

for all $x \in V$.

Note that if u is a solution to (1.1) then $u \in \Psi$.

Remark 4.2. For any $\psi \in \Psi$ and $v \in L^2(\Omega, C([0, T]; H))$,

$$\begin{aligned} \mathbb{E} \left[\int_0^t e^{-\int_0^s \rho(\psi_r) dr} \rho(\psi_s) |v_s|_H^2 ds \right] &\leq \mathbb{E} \sup_{s \in [0, t]} |v_s|_H^2 \int_0^t e^{-\int_0^s \rho(\psi_r) dr} \rho(\psi_s) ds \\ &= \mathbb{E} \sup_{s \in [0, t]} |v_s|_H^2 [1 - e^{-\int_0^t \rho(\psi_r) dr}] \leq \mathbb{E} \sup_{s \in [0, t]} |v_s|_H^2 < \infty. \end{aligned}$$

This remark justifies the existence of the bounded variation integrals appearing in the proof of uniqueness that follows.

Proof of Theorem 2.5. Consider two solutions u and \bar{u} of (1.1). Thus,

$$u_t - \bar{u}_t = \int_0^t (A_s(u_s) - A_s(\bar{u}_s)) ds + \sum_{j=1}^{\infty} \int_0^t (B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j \quad (4.12)$$

almost surely for all $t \in [0, T]$. Using the Itô's formula and the product rule one obtains

$$\begin{aligned} d \left(e^{-\int_0^t \rho(\bar{u}_s) ds} |u_t - \bar{u}_t|_H^2 \right) &= e^{-\int_0^t \rho(\bar{u}_s) ds} [d|u_t - \bar{u}_t|_H^2 - \rho(\bar{u}_t) |u_t - \bar{u}_t|_H^2 dt] \\ &= e^{-\int_0^t \rho(\bar{u}_s) ds} \left[\left(2 \langle A_t(u_t) - A_t(\bar{u}_t), u_t - \bar{u}_t \rangle + \sum_{j=1}^{\infty} |B_t^j(u_t) - B_t^j(\bar{u}_t)|_H^2 \right) dt \right. \\ &\quad \left. + \sum_{j=1}^{\infty} 2 \langle u_t - \bar{u}_t, B_t^j(u_t) - B_t^j(\bar{u}_t) \rangle dW_t^j - \rho(\bar{u}_t) |u_t - \bar{u}_t|_H^2 dt \right] \end{aligned}$$

almost surely for all $t \in [0, T]$. With Assumption A-2 one sees that, with $t_n := t \wedge \sigma_n$ and σ_n given by (4.3),

$$\begin{aligned} e^{-\int_0^{t_n} \rho(\bar{u}_s) ds} |u_{t_n} - \bar{u}_{t_n}|_H^2 \\ \leq 2 \sum_{j=1}^{\infty} \int_0^{t_n} e^{-\int_0^s \rho(\bar{u}_r) dr} (u_s - \bar{u}_s, B_s^j(u_s) - B_s^j(\bar{u}_s)) dW_s^j. \end{aligned}$$

Then,

$$\mathbb{E} [e^{-\int_0^{t_n} \rho(\bar{u}_s) ds} |u_{t_n} - \bar{u}_{t_n}|_H^2] \leq 0.$$

Letting $n \rightarrow \infty$ and using Fatou's lemma one concludes that for all $t \in [0, T]$ one has $\mathbb{P}(|u_t - \bar{u}_t|_H^2 = 0) = 1$. This, together with the continuity of $u - \bar{u}$ in H , concludes the proof. \square

5. A PRIORI ESTIMATES FOR GALERKIN DISCRETIZATION

Existence of solution to stochastic evolution equation (1.1) will now be shown using the Galerkin method. Consider a Galerkin scheme $(V_m)_{m \in \mathbb{N}}$ for V , i.e. for each $m \in \mathbb{N}$, V_m is an m -dimensional subspace of V such that $V_m \subset V_{m+1} \subset V$ and $\cup_{m \in \mathbb{N}} V_m$ is dense in V . Let $\{\phi_i : i = 1, 2, \dots, m\}$ be a basis of V_m . Assume that for each $m \in \mathbb{N}$, u_0^m is a V_m -valued \mathcal{F}_0 -measurable random variable satisfying

$$\sup_{m \in \mathbb{N}} \mathbb{E} |u_0^m|^{p_0} < \infty \text{ and } \mathbb{E} |u_0^m - u_0|_H^2 \rightarrow 0 \text{ as } m \rightarrow \infty.$$

It is always possible to obtain such an approximating sequence. For example, consider $\{\phi_i\}_{i \in \mathbb{N}} \subset V$ forming an orthonormal basis in H and for each $m \in \mathbb{N}$, take $u_0^m = \Pi_m u_0$ where $\Pi_m : H \rightarrow V_m$ are the projection operators.

For each $m \in \mathbb{N}$ and $\phi_i \in V_m$, $i = 1, 2, \dots, m$, consider the stochastic differential equation:

$$(u_t^m, \phi_i) = (u_0^m, \phi_i) + \int_0^t \langle A_s(u_s^m), \phi_i \rangle ds + \sum_{j=1}^m \int_0^t (\phi_i, B_s^j(u_s^m)) dW_s^j \quad (5.1)$$

almost surely for all $t \in [0, T]$. Using the results on solvability of stochastic differential equations in finite dimensional space (see, e.g., Theorem 3.1 in [8]), together with Assumptions A-1 to A-4 and Remark 2.2, there exists a unique adapted and continuous (and thus progressively measurable) V_m -valued process u^m satisfying (5.1).

Lemma 5.1 (A priori Estimates for Galerkin Discretization). *Suppose that u_t^m satisfies (5.1). Then under the Assumptions A-3 and A-4, one has, with C independent of m ,*

$$\sup_{t \in [0, T]} \mathbb{E} |u_t^m|_H^{p_0} + \mathbb{E} \int_0^T |u_t^m|_V^\alpha dt + \mathbb{E} \int_0^T |u_t^m|_H^{p_0-2} |u_t^m|_V^\alpha dt \leq C, \quad (5.2)$$

$$\mathbb{E} \sup_{t \in [0, T]} |u_t^m|_H^p \leq C, \quad (5.3)$$

with $p = 2$ in case $p_0 = 2$ (i.e. $\beta = 0$) and $p \in [2, p_0)$ if $p_0 > 2$,

$$\mathbb{E} \int_0^T |A_s(u_s^m)|_{V^*}^{\frac{\alpha}{\alpha-1}} ds \leq C, \quad (5.4)$$

$$\mathbb{E} \sum_{j=1}^\infty \int_0^T |B_s^j(u_s^m)|_H^2 ds \leq C. \quad (5.5)$$

Proof. Proof of (5.2) and (5.3) is almost a repetition of the proof of analogous results in Theorem 2.4. Indeed, for each $m \in \mathbb{N}$, one can define a sequence of stopping times

$$\sigma_n^m := \inf\{t \in [0, T] : |u_t^m|_H > n\} \wedge T$$

and repeat the steps of Theorem 2.4 by replacing u_t with u_t^m and σ_n with σ_n^m . There are two main points to be noted. The first is that the stochastic integral appearing on right-hand side of (4.1), with u_t replaced by u_t^m , is a local martingale for each $m \in \mathbb{N}$. Indeed, on a finite dimensional space, all norms are equivalent and hence

$$\mathbb{E} \int_0^{T \wedge \sigma_n^m} |u_t^m|_V^\alpha dt \leq C_m \mathbb{E} \int_0^{T \wedge \sigma_n^m} n^\alpha dt < \infty$$

with some constant C_m . The second point is that, since

$$\sup_{m \in \mathbb{N}} \mathbb{E} |u_0^m|^{p_0} < \infty,$$

one can take a constant independent of m to obtain (5.2) and (5.3).

The estimates (5.4) and (5.5) can be proved as below. One obtains from Assumption A-4, that

$$\begin{aligned} I &:= \mathbb{E} \int_0^T |A_s(u_s^m)|_{V^*}^{\frac{\alpha}{\alpha-1}} ds \leq \mathbb{E} \int_0^T (f_s + K|u_s^m|_V^\alpha)(1 + |u_s^m|_H^\beta) ds \\ &= \mathbb{E} \int_0^T f_s ds + \mathbb{E} \int_0^T f_s |u_s^m|_H^\beta ds + K \mathbb{E} \int_0^T |u_s^m|_V^\alpha ds \\ &\quad + K \mathbb{E} \int_0^T |u_s^m|_V^\alpha |u_s^m|_H^\beta ds. \end{aligned}$$

Using Young's inequality one can see that

$$f_s + f_s |u_s^m|_H^\beta \leq \frac{4}{p_0} f_s^{\frac{p_0}{2}} + \frac{p_0 - 2}{p_0} + \frac{p_0 - 2}{p_0} |u_s^m|_H^{\beta \frac{p_0}{p_0-2}}.$$

Moreover, $|u_s^m|_H^\beta \leq (1 + |u_s^m|_H)^{p_0-2}$, since $p_0 \geq \beta + 2$. Hence

$$\begin{aligned} I &\leq \frac{4}{p_0} \mathbb{E} \int_0^T f_s^{\frac{p_0}{2}} ds + \frac{p_0 - 2}{p_0} T + \frac{p_0 - 2}{p_0} \mathbb{E} \int_0^T |u_s^m|_H^{\beta \frac{p_0}{p_0-2}} ds \\ &\quad + K \mathbb{E} \int_0^T |u_s^m|_V^\alpha ds + K \mathbb{E} \int_0^T |u_s^m|_V^\alpha (1 + |u_s^m|_H)^{p_0-2} ds. \end{aligned}$$

Furthermore, applying Hölder's inequality,

$$\begin{aligned} I &\leq \frac{4}{p_0} \mathbb{E} \int_0^T f_s^{\frac{p_0}{2}} ds + \frac{p_0 - 2}{p_0} T + \frac{p_0 - 2}{p_0} T^{\frac{p_0-2-\beta}{p_0-2}} \left(\mathbb{E} \int_0^T |u_s^m|_H^{p_0} ds \right)^{\frac{\beta}{p_0-2}} \\ &\quad + (2^{p_0-3} + 1) K \mathbb{E} \int_0^T |u_s^m|_V^\alpha ds + 2^{p_0-3} K \int_0^T |u_s^m|_V^\alpha |u_s^m|_H^{p_0-2} ds \\ &\leq \frac{4}{p_0} \mathbb{E} \int_0^T f_s^{\frac{p_0}{2}} ds + \frac{p_0 - 2}{p_0} T + \frac{p_0 - 2}{p_0} T \sup_{0 \leq s \leq T} \mathbb{E} |u_s^m|_H^{p_0} \\ &\quad + (2^{p_0-3} + 1) K \mathbb{E} \int_0^T |u_s^m|_V^\alpha ds + 2^{p_0-3} K \int_0^T |u_s^m|_V^\alpha |u_s^m|_H^{p_0-2} ds, \end{aligned} \tag{5.6}$$

where one has used the fact $p_0 \geq \beta + 2$. By using (5.2) in (5.6), one obtains (5.4). Furthermore, by Remark 2.1, one gets

$$\begin{aligned} \mathbb{E} \int_0^T \sum_{j=1}^{\infty} |B_s^j(u_s^m)|_H^2 ds &\leq C \left[T + \mathbb{E} \int_0^T f_t^{\frac{p_0}{2}} ds + \mathbb{E} \int_0^T |u_s^m|_H^{p_0} ds \right. \\ &\quad \left. + \mathbb{E} \int_0^T |u_s^m|_V^\alpha ds + \mathbb{E} \int_0^T |u_s^m|_V^\alpha (1 + |u_s^m|_H)^{p_0-2} ds \right] \\ &\leq C \left[T + \mathbb{E} \int_0^T f_t^{\frac{p_0}{2}} ds + T \sup_{s \in [0, T]} \mathbb{E} |u_s^m|_H^{p_0} \right. \\ &\quad \left. + \mathbb{E} \int_0^T |u_s^m|_V^\alpha ds + \mathbb{E} \int_0^T |u_s^m|_V^\alpha |u_s^m|_H^{p_0-2} ds \right] \end{aligned}$$

and hence by using (5.2), one gets (5.5). \square

6. EXISTENCE OF SOLUTION

Having obtained the necessary a priori estimates, weakly convergent subsequences are extracted using the compactness argument. After that the local monotonicity condition is used to establish the existence of a solution to (1.1).

Lemma 6.1. *Let Assumptions A-2, A-3 and A-4 hold and assume that*

$$\sup_{m \in \mathbb{N}} \mathbb{E}|u_0^m|^{p_0} < \infty \text{ and } \mathbb{E}|u_0^m - u_0|_H^2 \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (6.1)$$

Then there is a subsequence $(m_k)_{k \in \mathbb{N}}$ and

- i) there exists a progressively measurable process $u \in L^\alpha((0, T) \times \Omega; V)$ such that*

$$u^{m_k} \rightharpoonup u \text{ in } L^\alpha((0, T) \times \Omega; V),$$

- ii) there exists a progressively measurable process $a \in L^{\frac{\alpha}{\alpha-1}}((0, T) \times \Omega; V^*)$ such that*

$$A(u^{m_k}) \rightharpoonup a \text{ in } L^{\frac{\alpha}{\alpha-1}}((0, T) \times \Omega; V^*),$$

- iii) there exists a progressively measurable process $b \in L^2((0, T) \times \Omega; l_2(H))$ such that*

$$B(u^{m_k}) \rightharpoonup b \text{ in } L^2((0, T) \times \Omega; l_2(H)).$$

Proof. The Banach spaces $L^\alpha((0, T) \times \Omega; V)$, $L^{\frac{\alpha}{\alpha-1}}((0, T) \times \Omega; V^*)$ and $L^2((0, T) \times \Omega; l_2(H))$ are reflexive. Thus, due to Lemma 5.1, there exists a subsequence m_k (see, e.g., Theorem 3.18 in [1]) such that

- (i) $u^{m_k} \rightharpoonup v$ in $L^\alpha((0, T) \times \Omega; V)$
- (ii) $A(u^{m_k}) \rightharpoonup a$ in $L^{\frac{\alpha}{\alpha-1}}((0, T) \times \Omega; V^*)$
- (iii) $(B^j(u^{m_k}))_{j=1}^k \rightharpoonup (b^j)_{j=1}^\infty$ in $L^2((0, T) \times \Omega; l_2(H))$.

□

Whilst not needed to prove results in this article, it is also possible to show that there is a subsequence of (m_k) , again denoted by m_k such that u^{m_k} converges weakly star to u in $L^p(\Omega; L^\infty(0, T; H))$. This is a consequence of Lemma 5.1 and Lemma 3.2.

Lemma 6.2. *Let Assumptions A-2, A-3 and A-4 together with (6.1) hold. Then*

- i) For $dt \times \mathbb{P}$ almost everywhere*

$$u_t = u_0 + \int_0^t a_s ds + \sum_{j=1}^\infty \int_0^t b_s^j dW_s^j$$

and moreover almost surely $u \in C([0, T]; H)$ and for all t

$$|u_t|_H^2 = |u_0|_H^2 + \int_0^t \left[2\langle a_s, u_s \rangle + \sum_{j=1}^\infty |b_s^j|_H^2 \right] ds + 2 \sum_{j=1}^\infty \int_0^t \langle u_s, b_s^j \rangle dW_s^j. \quad (6.2)$$

- ii) Finally, $u \in L^2(\Omega; C([0, T]; H))$.*

Proof. Using Itô's isometry, it can be shown that the stochastic integral is a bounded linear operator from $L^2((0, T) \times \Omega; l_2(H))$ to $L^2((0, T) \times \Omega; H)$ and hence maps a weakly convergent sequence to a weakly convergent sequence. Thus, one obtains

$$\sum_{j=1}^k \int_0^{\cdot} B_s^j(u_s^{m_k}) dW_s^j \rightharpoonup \sum_{j=1}^{\infty} \int_0^{\cdot} b_s^j dW_s^j$$

in $L^2([0, T] \times \Omega; H)$, i.e. for any $\psi \in L^2((0, T) \times \Omega; H)$,

$$\mathbb{E} \int_0^T \left(\sum_{j=1}^k \int_0^t B_s^j(u_s^{m_k}) dW_s^j, \psi(t) \right) dt \rightarrow \mathbb{E} \int_0^T \left(\sum_{j=1}^{\infty} \int_0^t b_s^j dW_s^j, \psi(t) \right) dt. \quad (6.3)$$

Similarly, using Holder's inequality it can be shown that the Bochner integral is a bounded linear operator from $L^{\frac{\alpha}{\alpha-1}}((0, T) \times \Omega; V^*)$ to $L^{\frac{\alpha}{\alpha-1}}((0, T) \times \Omega; V^*)$ and is thus continuous with respect to weak topologies. Therefore, for any $\psi \in L^{\alpha}((0, T) \times \Omega; V)$,

$$\mathbb{E} \int_0^T \left\langle \int_0^t A_s(u_s^{m_k}) ds, \psi(t) \right\rangle dt \rightarrow \mathbb{E} \int_0^T \left\langle \int_0^t a_s ds, \psi(t) \right\rangle dt. \quad (6.4)$$

Fix $n \in \mathbb{N}$. Then for any $\phi \in V_n$ and an adapted real valued process η_t bounded by a constant C , one has, for any $k \geq n$,

$$\begin{aligned} & \mathbb{E} \int_0^T \eta_t(u_t^{m_k}, \phi) dt \\ &= \mathbb{E} \int_0^T \eta_t \left((u_0^{m_k}, \phi) + \int_0^t \langle A_s(u_s^{m_k}), \phi \rangle ds + \sum_{j=1}^{\infty} \int_0^t (\phi, B_s^j(u_s^{m_k})) dW_s^j \right) dt. \end{aligned}$$

Taking the limit $k \rightarrow \infty$ and using (6.1), (6.3) and (6.4), one obtains

$$\begin{aligned} & \mathbb{E} \int_0^T \eta_t(v_t, \phi) dt \\ &= \mathbb{E} \int_0^T \eta_t \left((u_0, \phi) + \int_0^t \langle a_s, \phi \rangle ds + \sum_{j=1}^{\infty} \int_0^t (\phi, b_s^j) dW_s^j \right) dt \end{aligned}$$

with any $\phi \in V_n$ and any adapted real valued process η_t bounded by a constant C . Since $\cup_{n \in \mathbb{N}} V_n$ is dense in V , one obtains

$$v_t = u_0 + \int_0^t a_s ds + \sum_{j=1}^{\infty} \int_0^t b_s^j dW_s^j \quad (6.5)$$

$dt \times \mathbb{P}$ almost everywhere. Now, using Theorem 3.2 on Itô's formula from [8], there exists an H -valued continuous modification u of v which is equal to the right hand side of (6.5) almost surely for all $t \in [0, T]$. Moreover (6.2) holds almost surely for all $t \in [0, T]$. This completes the proof of part (i) of the lemma. It remains to prove part (ii) of the lemma. To that end, consider the sequence of stopping times σ_n defined for each $n \in \mathbb{N}$ by

$$\sigma_n := \inf\{t \in [0, T] : |u_t|_H > n\} \wedge T.$$

From the Burkholder–Davis–Gundy inequality, one obtains

$$\mathbb{E} \sup_{t \in [0, T]} \left| \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, b_s^j) dW_s^j \right| \leq 4 \mathbb{E} \left(\sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |(u_s, b_s^j)|_H^2 ds \right)^{\frac{1}{2}}.$$

Using Cauchy–Schwartz’s and Young’s inequalities leads to

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left| \sum_{j=1}^{\infty} \int_0^{t \wedge \sigma_n} (u_s, b_s^j) dW_s^j \right| &\leq 4 \mathbb{E} \left(\sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |u_s|_H^2 |b_s^j|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq 4 \mathbb{E} \left(\sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 \sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |b_s^j|_H^2 ds \right)^{\frac{1}{2}} \\ &\leq \epsilon \mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 + C \mathbb{E} \sum_{j=1}^{\infty} \int_0^{T \wedge \sigma_n} |b_s^j|_H^2 ds. \end{aligned} \quad (6.6)$$

Taking supremum and then expectation in (6.2) and using Hölder’s inequality along with (6.6), one obtains

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 &\leq \mathbb{E} |u_0|_H^2 + 2 \left(\mathbb{E} \int_0^T |a_s|^{\frac{\alpha}{\alpha-1}} ds \right)^{\frac{\alpha-1}{\alpha}} \left(\mathbb{E} \int_0^T |u_s|_V^\alpha ds \right)^{\frac{1}{\alpha}} \\ &\quad + \epsilon \mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 + C \mathbb{E} \sum_{j=1}^{\infty} \int_0^T |b_s^j|_H^2 ds \end{aligned}$$

which on choosing ϵ small enough gives

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} |u_{t \wedge \sigma_n}|_H^2 &\leq C \left[\mathbb{E} |u_0|_H^2 + \left(\mathbb{E} \int_0^T |a_s|^{\frac{\alpha}{\alpha-1}} ds \right)^{\frac{\alpha-1}{\alpha}} \left(\mathbb{E} \int_0^T |u_s|_V^\alpha ds \right)^{\frac{1}{\alpha}} + \mathbb{E} \sum_{j=1}^{\infty} \int_0^T |b_s^j|_H^2 ds \right]. \end{aligned}$$

Finally taking $n \rightarrow \infty$ and using Fatou’s lemma, one obtains

$$\mathbb{E} \sup_{t \in [0, T]} |u_t|_H^2 < \infty.$$

This concludes the proof. \square

From now onwards, the processes v and u will be denoted by u for notational convenience. In order to prove that the process u is the solution of equation (1.1), it remains to show that $dt \times \mathbb{P}$ almost everywhere $A(v) = a$ and $B^j(v) = b^j$ for all $j \in \mathbb{N}$. Recall that Ψ and ρ were given in Definition 4.1.

Proof of Theorem 2.6. For $\psi \in L^\alpha((0, T) \times \Omega; V) \cap \Psi \cap L^2(\Omega; C([0, T]; H))$, using the product rule and Itô’s formula one obtains

$$\begin{aligned} &\mathbb{E} \left(e^{-\int_0^t \rho(\psi_s) ds} |u_t|_H^2 \right) - \mathbb{E} (|u_0|_H^2) \\ &= \mathbb{E} \left[\int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2 \langle a_s, u_s \rangle + \sum_{j=1}^{\infty} |b_s^j|_H^2 - \rho(\psi_s) |u_s|_H^2 \right) ds \right] \end{aligned} \quad (6.7)$$

and

$$\begin{aligned}
& \mathbb{E}(e^{-\int_0^t \rho(\psi_s) ds} |u_t^{m_k}|_H^2) - \mathbb{E}(|u_0^{m_k}|_H^2) \\
&= \mathbb{E} \left[\int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2\langle A_s(u_s^{m_k}), u_s^{m_k} \rangle \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{\infty} |B_s^j(u_s^{m_k})|_H^2 - \rho(\psi_s) |u_s^{m_k}|_H^2 \right) ds \right] \tag{6.8}
\end{aligned}$$

for all $t \in [0, T]$. Note that in view of Remark 4.2, all the integrals are well defined in what follows. Moreover,

$$\begin{aligned}
& \mathbb{E} \left[\int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2\langle A_s(u_s^{m_k}), u_s^{m_k} \rangle + \sum_{j=1}^{\infty} |B_s^j(u_s^{m_k})|_H^2 - \rho(\psi_s) |u_s^{m_k}|_H^2 \right) ds \right] \\
&= \mathbb{E} \left[\int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2\langle A_s(u_s^{m_k}) - A_s(\psi_s), u_s^{m_k} - \psi_s \rangle + 2\langle A_s(\psi_s), u_s^{m_k} \rangle \right. \right. \\
&\quad \left. \left. + 2\langle A_s(u_s^{m_k}) - A_s(\psi_s), \psi_s \rangle + \sum_{j=1}^{\infty} |B_s^j(u_s^{m_k}) - B_s^j(\psi_s)|_H^2 - \sum_{j=1}^{\infty} |B_s^j(\psi_s)|_H^2 \right. \right. \\
&\quad \left. \left. + 2 \sum_{j=1}^{\infty} (B_s^j(u_s^{m_k}), B_s^j(\psi_s)) - \rho(\psi_s) [|u_s^{m_k} - \psi_s|_H^2 - |\psi_s|_H^2 + 2(u_s^{m_k}, \psi_s)] \right) ds \right].
\end{aligned}$$

Now one can apply the local monotonicity Assumption A-2 to see that

$$\begin{aligned}
& \mathbb{E}(e^{-\int_0^t \rho(\psi_s) ds} |u_t^{m_k}|_H^2) - \mathbb{E}(|u_0^{m_k}|_H^2) \\
&\leq \mathbb{E} \left[\int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2\langle A_s(\psi_s), u_s^{m_k} \rangle + 2\langle A_s(u_s^{m_k}) - A_s(\psi_s), \psi_s \rangle \right. \right. \\
&\quad \left. \left. - \sum_{j=1}^{\infty} |B_s^j(\psi_s)|_H^2 + 2 \sum_{j=1}^{\infty} (B_s^j(u_s^{m_k}), B_s^j(\psi_s)) + \rho(\psi_s) [|\psi_s|_H^2 - 2(u_s^{m_k}, \psi_s)] \right) ds \right]
\end{aligned}$$

Integrating over t from 0 to T , letting $k \rightarrow \infty$ and using the weak lower semicontinuity of the norm one obtains

$$\begin{aligned}
& \mathbb{E} \left[\int_0^T (e^{-\int_0^t \rho(\psi_s) ds} |u_t|_H^2 - |u_0|_H^2) dt \right] \\
&\leq \liminf_{k \rightarrow \infty} \mathbb{E} \left[\int_0^T (e^{-\int_0^t \rho(\psi_s) ds} |u_t^{m_k}|_H^2 - |u_0^{m_k}|_H^2) dt \right] \\
&\leq \mathbb{E} \left[\int_0^T \int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2\langle A_s(\psi_s), u_s \rangle + 2\langle a_s - A_s(\psi_s), \psi_s \rangle - \sum_{j=1}^{\infty} |B_s^j(\psi_s)|_H^2 \right. \right. \\
&\quad \left. \left. + 2 \sum_{j=1}^{\infty} (b_s^j, B_s^j(\psi_s)) + \rho(\psi_s) [|\psi_s|_H^2 - 2(u_s, \psi_s)] \right) ds dt \right]. \tag{6.9}
\end{aligned}$$

Integrating from 0 to T in (6.7) and combining this with (6.9) leads to

$$\mathbb{E} \left[\int_0^T \int_0^t e^{-\int_0^s \rho(\psi_r) dr} \left(2 \langle a_s - A_s(\psi_s), u_s - \psi_s \rangle + \sum_{j=1}^{\infty} |B_s^j(\psi_s) - b_s^j|_H^2 - \rho(\psi_s) |u_s - \psi_s|_H^2 \right) ds dt \right] \leq 0. \quad (6.10)$$

Further, using the Definition 4.1 and Lemma 6.1,

$$u \in L^\alpha((0, T) \times \Omega; V) \cap \Psi \cap L^2(\Omega; C([0, T]; H)).$$

Taking $\psi = u$ in (6.10), one obtains that $B^j(u) = b^j$ for all $j \in \mathbb{N}$. Let $\eta \in L^\infty((0, T) \times \Omega; \mathbb{R})$, $\phi \in V$, $\epsilon \in (0, 1)$ and let $\psi = u - \epsilon \eta \phi$. Then from (6.10) one obtains that

$$\mathbb{E} \left[\int_0^T \int_0^t e^{-\int_0^s \rho(u_r - \epsilon \eta_r \phi) dr} \left(2 \epsilon \langle a_s - A_s(u_s - \epsilon \eta_s \phi), \eta_s \phi \rangle - \epsilon^2 \rho(u_s - \epsilon \eta_s \phi) |\eta_s \phi|_H^2 \right) ds dt \right] \leq 0. \quad (6.11)$$

Dividing by ϵ , letting $\epsilon \rightarrow 0$, using Lebesgue dominated convergence theorem and Assumption A-1 leads to

$$\mathbb{E} \left[\int_0^T \int_0^t e^{-\int_0^s \rho(u_r) dr} 2 \eta_s \langle a_s - A_s(u_s), \phi \rangle ds dt \right] \leq 0.$$

Since this holds for any $\eta \in L^\infty((0, T) \times \Omega; \mathbb{R})$ and $\phi \in V$, one gets that $A(u) = a$ which concludes the proof. \square

7. EXAMPLES

In this section, some examples of stochastic evolution equations are presented which fit in the framework of this article and yet do not satisfy the assumptions of [8, 13].

Throughout the section, \mathbb{R}^d denotes a d -dimensional Euclidean space. Let $\mathcal{D} \subseteq \mathbb{R}^d$ be an open bounded domain with smooth boundary. Then for any $p \geq 1$, $L^p(\mathcal{D})$ is the Lebesgue space of equivalence classes of real valued measurable functions u defined on \mathcal{D} such that the norm

$$|u|_{L^p} := \left(\int_{\mathcal{D}} |u(x)|^p dx \right)^{\frac{1}{p}}$$

is finite. For $i \in \{1, 2, \dots, d\}$, let D_i denote the distributional derivative along the i -th coordinate in \mathbb{R}^d . Further, let $\nabla := (D_1, D_2, \dots, D_d)$ denote the gradient. Finally, $W^{1,2}(\mathcal{D})$ is the Sobolev space of real valued functions u , defined on \mathcal{D} , such that the norm

$$|u|_{1,2} := \left(\int_{\mathcal{D}} (|u(x)|^2 + |\nabla u(x)|^2) dx \right)^{\frac{1}{2}}$$

is finite.

Let $C_0^\infty(\mathcal{D})$ be the space of smooth functions with compact support in \mathcal{D} . Then, the closure of $C_0^\infty(\mathcal{D})$ in $L^2(\mathcal{D})$ with respect to the norm $|\cdot|_{1,2}$

is denoted by $W_0^{1,2}(\mathcal{D})$. Friederichs' inequality (see, e.g. Theorem 1.32 in [14]) implies that the norm

$$|u|_{W_0^{1,2}} := \left(\int_{\mathcal{D}} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}$$

is equivalent to $|u|_{1,2}$ and this equivalent norm $|u|_{W_0^{1,2}}$ will be used throughout this section. Moreover, let $W^{-1,2}(\mathcal{D})$ denote the dual of $W_0^{1,2}(\mathcal{D})$ and let $|\cdot|_{W^{-1,2}}$ be the norm on this dual space. It is well known that

$$W_0^{1,2}(\mathcal{D}) \hookrightarrow L^2(\mathcal{D}) \equiv (L^2(\mathcal{D}))^* \hookrightarrow W^{-1,2}(\mathcal{D}),$$

where \hookrightarrow denotes continuous and dense embeddings, is a Gelfand triple. Finally, define $\Delta : W_0^{1,2}(\mathcal{D}) \rightarrow W^{-1,2}(\mathcal{D})$ by

$$\langle \Delta u, v \rangle := - \int_{\mathcal{D}} \nabla u(x) \nabla v(x) dx \quad \forall v \in W_0^{1,2}(\mathcal{D}).$$

Clearly

$$|\Delta u|_{W^{-1,2}} \leq |u|_{W_0^{1,2}} \quad (7.1)$$

and so the operator is linear and bounded.

The following consequence of Gagliardo–Nirenberg inequality (see, e.g., Theorem 1.24 in [14]) will be needed in the examples presented below. If $d = 2$, then there exists a constant C such that

$$|u|_{L^4} \leq C |u|_{L^2}^{\frac{1}{2}} |u|_{W_0^{1,2}}^{\frac{1}{2}}. \quad (7.2)$$

Further, if $d = 1$, then there exists a constant C such that

$$|u|_{L^4} \leq C |u|_{L^2}^{\frac{3}{4}} |u|_{W_0^{1,2}}^{\frac{1}{4}} \leq C |u|_{L^2}^{\frac{1}{2}} |u|_{W_0^{1,2}}^{\frac{1}{2}}.$$

Example 7.1 (Semi-linear equation). Let γ be a constant. For $i = 1, 2, \dots, d$, let $g_i : \mathbb{R} \rightarrow \mathbb{R}$ be bounded and Lipschitz continuous and $h_i : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz continuous. Let there be constants $r, s \geq 0$ and a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x)| \leq K(1+|x|^r) \quad \text{and} \quad (f(x)-f(y))(x-y) \leq K(1+|y|^s)|x-y|^2 \quad \forall x, y \in \mathbb{R}.$$

Consider the stochastic partial differential equation

$$du_t = (\Delta u_t + g(u_t) \nabla u_t + f(u_t)) dt + (\gamma \nabla u_t + h(u_t)) dW_t \quad \text{on } (0, T) \times \mathcal{D}, \quad (7.3)$$

where $u_t = 0$ on $\partial \mathcal{D}$, u_0 is a given \mathcal{F}_0 -measurable random variable and Δ is the Laplace operator. It will now be shown that such an equation, in its weak form, fits the assumptions of the present article for certain values of r, s, γ and with $d = 1$ or 2 .

Let $A : W_0^{1,2}(\mathcal{D}) \rightarrow W^{-1,2}(\mathcal{D})$ and $B^i : W_0^{1,2}(\mathcal{D}) \rightarrow L^2(\mathcal{D})$ be given by

$$A(u) := \Delta u + g(u) \nabla u + f(u) \quad \text{and} \quad B^i(u) := \gamma D_i u + h_i(u)$$

for $i = 1, 2, \dots, d$. The next step is to show that these operators satisfy the Assumptions A-1 to A-4. One immediately notices that A-1 holds, in particular, since g and f are continuous.

One now wishes to verify the local monotonicity condition. By using the assumptions imposed on f and g one can see for $u, v \in W_0^{1,2}(\mathcal{D})$, upon application of Hölder's inequality, that

$$\begin{aligned} & \langle A(u) - A(v), u - v \rangle \\ &= -|u - v|_{W_0^{1,2}}^2 + \langle g(u)\nabla(u - v), u - v \rangle + \langle (\nabla v)(g(u) - g(v)), u - v \rangle \\ & \quad + \langle f(u) - f(v), u - v \rangle \\ &\leq -|u - v|_{W_0^{1,2}}^2 + C|u - v|_{W_0^{1,2}}|u - v|_{L^2} + C|v|_{W_0^{1,2}}|u - v|_{L^4}^2 \\ & \quad + C|u - v|_{L^2}^2 + C|v|_{L^{2s}}^s|u - v|_{L^4}^2. \end{aligned}$$

Then (7.2) implies that

$$\begin{aligned} & \langle A(u) - A(v), u - v \rangle \\ &\leq -|u - v|_{W_0^{1,2}}^2 + C|u - v|_{W_0^{1,2}}|u - v|_{L^2} + C|v|_{W_0^{1,2}}|u - v|_{L^2}|u - v|_{W_0^{1,2}} \\ & \quad + C|u - v|_{L^2}^2 + C|v|_{L^{2s}}^s|u - v|_{W_0^{1,2}}|u - v|_{L^2}. \end{aligned}$$

Young's inequality with some $\epsilon > 0$ finally leads to

$$\begin{aligned} & \langle A(u) - A(v), u - v \rangle \\ &\leq (\epsilon - 1)|u - v|_{W_0^{1,2}}^2 + C(1 + |v|_{W_0^{1,2}}^2 + |v|_{L^{2s}}^{2s})|u - v|_{L^2}^2. \end{aligned} \quad (7.4)$$

Moreover,

$$\sum_{i=1}^d |B^i(u) - B^i(v)|_{L^2}^2 \leq 2\gamma^2|u - v|_{W_0^{1,2}}^2 + C|u - v|_{L^2}^2$$

and so with $s = 2$, using (7.2) once again, one obtains

$$\begin{aligned} & 2\langle A(u) - A(v), u - v \rangle + \sum_{i=1}^d |B^i(u) - B^i(v)|_{L^2}^2 \\ &\leq (2\epsilon + 2\gamma^2 - 2)|u - v|_{W_0^{1,2}}^2 + C(1 + |v|_{W_0^{1,2}}^2 + |v|_{L^2}^2|v|_{W_0^{1,2}}^2)|u - v|_{L^2}^2. \end{aligned}$$

If $\gamma \in (-1, 1)$, then one can get that for some $\theta > 0$,

$$\begin{aligned} & 2\langle A(u) - A(v), u - v \rangle + \sum_{i=1}^d |B^i(u) - B^i(v)|_{L^2}^2 + \theta|u - v|_{W_0^{1,2}}^2 \\ &\leq C(1 + |v|_{W_0^{1,2}}^2)(1 + |v|_{L^2}^2)|u - v|_{L^2}^2, \end{aligned}$$

for all $u, v \in W_0^{1,2}(\mathcal{D})$. Hence Assumption A-2 is satisfied with $\alpha := 2$ and $\beta := 2$.

The next condition that ought to be verified is coercivity. Taking $v = 0$ in (7.4), one obtains for all $u \in W_0^{1,2}(\mathcal{D})$

$$\langle A(u), u \rangle \leq (\epsilon - 1)|u|_{W_0^{1,2}}^2 + C|u|_{L^2}^2$$

which implies, together with the assumptions on h , that

$$\begin{aligned} 2\langle A(u), u \rangle + (p_0 - 1) \sum_{i=1}^d |B^i(u)|_{L^2}^2 \\ \leq (\epsilon + 2\gamma^2(p_0 - 1) - 2) |u|_{W_0^{1,2}}^2 + C(1 + |u|_{L^2}^2). \end{aligned}$$

One can now take $p_0 := 4$ and see that if $\gamma^2 < 1/3$, then Assumption A-3 holds with $\theta := 2 - \epsilon - 6\gamma^2$ for $\epsilon > 0$ sufficiently small.

Finally one wishes to verify the growth condition. Using the boundedness of g and Hölder's inequality one obtains, for $u \in W_0^{1,2}(\mathcal{D})$, that

$$|g(u)\nabla u|_{W^{-1,2}} \leq C|u|_{W_0^{1,2}}.$$

Moreover, due to Hölder's inequality, one gets that for any $1 \leq q < \infty$ and $u, v \in W_0^{1,2}(\mathcal{D})$

$$\langle f(u), v \rangle \leq C|v|_{L^2} + C|u|_{L^{\frac{r}{r-\frac{q}{q-1}}}}^r |v|_{L^q} \leq C|v|_{L^2} + C|u|_{L^{\frac{r}{r-\frac{q}{q-1}}}}^r |v|_{W_0^{1,2}},$$

where the last inequality is consequence of the Sobolev embedding and the fact that $d = 1$ or 2 . Hence, with $q = 6$ and $r \leq \frac{7}{3}$, one obtains with (7.2), that

$$|f(u)|_{W^{-1,2}} \leq C \left(1 + |u|_{L^{\frac{r}{3r}}}^r \right) \leq C \left(1 + |u|_{L^2}^{\frac{4}{3}} |u|_{L^6} \right),$$

where the last inequality follows from interpolation between spaces of integrable functions, see e.g. [14, Theorem 1.24]. Finally, using the Sobolev embedding again, one can see that

$$|A(u)|_{W^{-1,2}}^2 \leq C \left(1 + |u|_{W_0^{1,2}}^2 \right) (1 + |u|_{L^2}^2)$$

thus Assumption A-4 is satisfied with $\alpha = 2$, $\beta = 2$.

If $d = 1$ or 2 , $\alpha = 2$, $\beta = 2$, $p_0 = 4$, $\gamma^2 < 1/3$ and $u_0 \in L^4(\Omega; L^2(\mathcal{D}))$ is \mathcal{F}_0 -measurable then, in view of Theorems 2.4, 2.5 and 2.6, one can conclude that equation (7.3) has a unique solution and moreover for any $p < 4$ one has

$$\mathbb{E} \left(\sup_{t \in [0, T]} |u_t|_{L^2}^p + \int_0^T |u_t|_{W_0^{1,2}}^2 dt \right) < C (1 + \mathbb{E}|u_0|^4).$$

Example 7.2 (Stochastic Burgers equation). Let $d = 1$ and $\mathcal{D} = (0, 1)$. Let $\gamma \in (-\sqrt{1/3}, \sqrt{1/3})$ be a constant and let $h : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz continuous. Consider the stochastic partial differential equation

$$du_t = \left(\Delta u_t + u_t Du_t \right) dt + (\gamma Du_t + h(u_t)) dW_t \text{ on } (0, T) \times \mathcal{D}, \quad (7.5)$$

where $u_t = 0$ on $\partial\mathcal{D}$ and an $L^2(\mathcal{D})$ -valued, \mathcal{F}_0 -measurable u_0 is a given initial condition. Weak formulation of this equation can be interpreted as a stochastic evolution equation as follows.

Define $A : W_0^{1,2}(\mathcal{D}) \rightarrow W^{-1,2}(\mathcal{D})$ and $B : W_0^{1,2}(\mathcal{D}) \rightarrow L^2(\mathcal{D})$ as

$$A(u) := \Delta u + u Du \quad \text{and} \quad B(u) := \gamma Du + h(u).$$

Note that Assumption A-1 is satisfied following the same arguments as in Example 7.1. Next, one would like to check the local monotonicity assumption. Note that, if $u, v \in W_0^{1,2}(\mathcal{D})$, then

$$\frac{1}{2}D[u^2 - v^2] = uDu - vDv$$

and so using integration by parts

$$\langle uDu - vDv, u - v \rangle = -\frac{1}{2}\langle u^2 - v^2, D(u - v) \rangle.$$

Thus,

$$\begin{aligned} & \langle A(u) - A(v), u - v \rangle \\ &= -|u - v|_{W_0^{1,2}}^2 - \frac{1}{2}\langle (u - v)^2, D(u - v) \rangle - \langle v(u - v), D(u - v) \rangle. \end{aligned}$$

But using integration by parts again we see that $\langle (u - v)^2, D(u - v) \rangle = 0$ and so

$$\langle A(u) - A(v), u - v \rangle = -|u - v|_{W_0^{1,2}}^2 - \langle v(u - v), D(u - v) \rangle.$$

So from Hölder's inequality one observes that

$$\langle A(u) - A(v), u - v \rangle \leq -|u - v|_{W_0^{1,2}}^2 + |v|_{L^4}|u - v|_{L^4}|u - v|_{W_0^{1,2}}$$

and thus Gagliardo–Nirenberg inequality, see (7.2), and Young's inequality imply that for any $\epsilon > 0$

$$\langle A(u) - A(v), u - v \rangle \leq -|u - v|_{W_0^{1,2}}^2 + \epsilon|u - v|_{W_0^{1,2}}^2 + C|v|_{L^2}^2|v|_{W_0^{1,2}}^2|u - v|_{L^2}^2. \quad (7.6)$$

This, along with Lipschitz continuity of h , gives

$$\begin{aligned} & 2\langle A(u) - A(v), u - v \rangle + |B(u) - B(v)|_{L^2}^2 \\ & \leq (-2 + 2\epsilon + 2\gamma^2)|u - v|_{W_0^{1,2}}^2 + C(1 + |v|_{L^2}^2)(1 + |v|_{W_0^{1,2}}^2)|u - v|_{L^2}^2 \end{aligned}$$

for all $u, v \in W_0^{1,2}(\mathcal{D})$. As $\gamma^2 \in (0, 1/3)$ one can take $\epsilon > 0$ sufficiently small so that $-1 + \epsilon + \gamma^2 < 0$ and hence Assumption A-2 is satisfied with $\alpha := 2$ and $\beta := 2$.

The next step is to show that the coercivity assumption holds with $p_0 = 4$. Indeed, substituting $v = 0$ in (7.6), one obtains

$$\langle A(u), u \rangle \leq (-1 + \epsilon)|u|_{W_0^{1,2}}^2$$

which along with linear growth of h implies that

$$2\langle A(u), u \rangle + 3|B(u)|_{L^2}^2 \leq (-2 + 2\epsilon + 6\gamma^2)|u|_{W_0^{1,2}}^2 + C(1 + |u|_{L^2}^2).$$

Note that since $\gamma^2 \in (0, 1/3)$ one can take $\epsilon > 0$ sufficiently small so that $\theta := 2 - 2\epsilon - 6\gamma^2 > 0$. Then with $f := C$, Assumption A-3 holds.

Finally, one should verify the growth assumption on A . Using integration by parts, Hölder's inequality and (7.2) one obtains for $u, v \in W_0^{1,2}(\mathcal{D})$,

$$\langle uDu, v \rangle = -\frac{1}{2}\langle u, Dv \rangle \leq \frac{1}{2}|u|_{L^4}^2|v|_{W_0^{1,2}} \leq C|u|_{L^2}|u|_{W_0^{1,2}}|v|_{W_0^{1,2}}$$

which then implies that

$$|uD u|_{W^{-1,2}} \leq C|u|_{L^2}|u|_{W_0^{1,2}}.$$

Hence using (7.1), one obtains for all $u \in W_0^{1,2}(\mathcal{D})$

$$|A(u)|_{W^{-1,2}}^2 \leq C|u|_{W_0^{1,2}}^2(1 + |u|_{L^2}^2)$$

proving that Assumption A-4 is satisfied for $\alpha = 2, \beta = 2$ and $f = C$.

Thus, in view of Theorems 2.4, 2.5 and 2.6, if $u_0 \in L^4(\Omega; L^2(\mathcal{D}))$, then equation (7.5) has a unique solution $(u_t)_{t \in [0, T]}$ and for any $p < 4$

$$\mathbb{E} \left(\sup_{t \in [0, T]} |u_t|_{L^2}^p + \int_0^T |u_t|_{W_0^{1,2}}^2 dt \right) < C(1 + \mathbb{E}|u_0|_{L^2}^4),$$

where we recall in particular that C depends on T .

In the previous two examples the range of values γ may take is restricted. This is not surprising in view of known results for linear stochastic partial differential equations where the “stochastic parabolicity” condition is needed. To see how this arises, consider the initial value problem

$$dv_t = (1 - \frac{1}{2}\gamma^2)\Delta v_t \text{ on } (0, T) \times \mathbb{R}^d$$

with $v_0 \in L^2(\mathbb{R}^d)$ given as an initial value. This is well-posed if $(1 - \frac{1}{2}\gamma^2) > 0$. Let $u_t(x) := v(t, x + \gamma W_t)$, where W is \mathbb{R} -valued Wiener process. Itô's formula implies that

$$du_t = \Delta u_t dt + \sum_{i=1}^d \gamma D_i u_t dW_t, \text{ on } (0, T) \times \mathbb{R}^d, \quad u_0 = v_0.$$

Hence one can only reasonably expect this stochastic partial differential equation to be well-posed if $(1 - \frac{1}{2}\gamma^2) > 0$.

On the other hand, one can see that the range of values of γ one may take, so that Assumption A-3 is satisfied, depends on p_0 . This may seem surprising in view of results in Krylov [7] on L^p -theory for stochastic partial differential equations. The following example, which is not covered in [7], from Brzeźniak and Veraar [3], explores this question further.

Example 7.3. Consider the stochastic partial differential equation

$$du_t = \Delta u_t dt + 2\gamma(-\Delta)^{\frac{1}{2}} u_t dW_t \text{ on } (0, T) \times \mathbb{T}, \quad (7.7)$$

where \mathbb{T} is the one-dimensional torus $\mathbb{R}/(2\pi\mathbb{Z})$, $\gamma \in \mathbb{R}$ is a constant and \mathcal{F}_0 -measurable u_0 is a given initial condition.

For $\gamma^2 \in (0, 1/2)$ and $u_0 \in L^2(\Omega; L^2(\mathbb{T}))$ the results in Krylov and Rozovskii [8] imply existence and uniqueness of the solution to (7.7) and moreover the solution satisfies

$$\mathbb{E} \sup_{t \in [0, T]} |u_t|_{L^2(\mathbb{T})}^2 < C\mathbb{E}(1 + |u_0|_{L^2(\mathbb{T})}^2).$$

On the other hand Brzeźniak and Veraar [3] have shown that if

$$2\gamma^2(p - 1) > 1,$$

then the problem (7.7) is not well-posed in $L^p((0, T) \times \Omega; L^2(\mathbb{T}))$. It will be shown that this example fits in the framework considered in this paper and that the coercivity condition, Assumption A-3, is satisfied as long as

$$2\gamma^2(p_0 - 1) < 1. \quad (7.8)$$

This shows that the coercivity condition in this paper is sharp, since (7.7) is ill-posed as soon as Assumption A-3 does not hold.

Let the space $L^2(\mathbb{T})$ denote the Lebesgue space of equivalence classes of \mathbb{C} -valued measurable functions u defined on any interval of length 2π , which are 2π -periodic and the norm

$$|u|_{L^2(\mathbb{T})} := \left(\int_{\mathbb{T}} |u(x)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

Further, $W^{1,2}(\mathbb{T})$ denotes the closure of $C^\infty(\mathbb{T})$, the space of smooth functions, in $L^2(\mathbb{T})$ with respect to the norm $|\cdot|_{1,2}$ given by

$$|u|_{1,2} := \left(\int_{\mathbb{T}} (|u(x)|^2 + |Du(x)|^2) dx \right)^{\frac{1}{2}}.$$

Let $\mathcal{F} : L^2(\mathbb{T}) \rightarrow l^2(\mathbb{Z})$ be the Fourier transform given by

$$\mathcal{F}u := (\hat{u}_k)_{k \in \mathbb{Z}} \text{ with } \hat{u}_k = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} u(x) e^{-ikx} dx$$

and $\mathcal{F}^{-1} : l^2(\mathbb{Z}) \rightarrow L^2(\mathbb{T})$ its inverse which is given by

$$\mathcal{F}^{-1}(\hat{u}_k)_{k \in \mathbb{Z}} =: u \text{ with } u(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \hat{u}(k) e^{ikx}.$$

For $u \in W^{1,2}(\mathbb{T})$, one has

$$|u|_{W^{1,2}(\mathbb{T})}^2 = |\mathcal{F}u|_{l^2(\mathbb{Z})}^2 + |\mathcal{F}(Du)|_{l^2(\mathbb{Z})}^2, \quad \text{since } |u|_{L^2(\mathbb{T})}^2 = |\mathcal{F}u|_{l^2(\mathbb{Z})}^2. \quad (7.9)$$

Furthermore, for each $k \in \mathbb{Z}$,

$$[\mathcal{F}(Du)](k) = ik(\mathcal{F}u)(k). \quad (7.10)$$

Consider the operator $(-\Delta)^{\frac{1}{2}} : W^{1,2}(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ defined by

$$(-\Delta)^{\frac{1}{2}} u := \mathcal{F}^{-1} \left((|k|(\mathcal{F}u)(k))_{k \in \mathbb{Z}} \right)$$

and the operators $A : W^{1,2}(\mathbb{T}) \rightarrow W^{-1,2}(\mathbb{T})$ and $B : W^{1,2}(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ defined by

$$A(u) = \Delta u \text{ and } B(u) = 2\gamma(-\Delta)^{\frac{1}{2}} u.$$

It will be shown that these satisfy Assumptions A-1 to A-4. Using the arguments given in Example 1, the operator A satisfies Assumptions A-1 and A-4 with $\alpha = 2$, $\beta = 0$, $p_0 = 2$ and $L = 0$. Then, using (7.9) and (7.10), one obtains

$$\begin{aligned} 2\langle A(u) - A(v), u - v \rangle + |B(u) - B(v)|_{L^2(\mathbb{T})}^2 \\ = (-2 + 4\gamma^2) \sum_{k \in \mathbb{Z}} k^2 |(\mathcal{F}u)(k) - (\mathcal{F}v)(k)|^2 \leq 0 \end{aligned}$$

provided $2\gamma^2 \leq 1$. Hence operators A and B satisfy Assumption A-2 if $2\gamma^2 \leq 1$. Furthermore, for any $\theta > 0$ and $p_0 \geq 2$, one obtains

$$\begin{aligned} & 2\langle A(u), u \rangle + (p_0 - 1)\|Bu\|_{L^2(\mathbb{T})}^2 + \theta\|u\|_{W^{1,2}(\mathbb{T})}^2 \\ &= (4\gamma^2(p_0 - 1) - 2 + \theta) \sum_{k \in \mathbb{Z}} k^2 |(\mathcal{F}u)(k)|^2 + \theta\|u\|_{L^2}^2. \end{aligned}$$

Note that there is $\theta > 0$ such that $(4\gamma^2(p_0 - 1) - 2 + \theta) \leq 0$ if and only if $2\gamma^2(p_0 - 1) < 1$. Hence Assumption A-3 holds if and only if (7.8) holds.

Thus from Theorem 2.4 one can see that the solution satisfies

$$\mathbb{E} \sup_{t \in [0, T]} \|u_t\|_{L^2(\mathbb{T})}^p < C\mathbb{E}\left(1 + \|u_0\|_{L^2(\mathbb{T})}^{p_0}\right)$$

for $p \in [2, p_0)$ if $p_0 > 2$ and for $p = 2$ otherwise.

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